. Name : Farah Ajeeb

. ID : g00077394

Homework 8

Another Solution to Q1.

Note that |A^T| = |-A| = (-1)^{2019}|A| = -|A|.
But |A^T| = |A|.
Hence |A^T| = |A| = -|A|. Thus 2|A| = 0. Hence |A| = 0.
Therefore A is not invertible (singular)

* Question 1: Let A be a skew-symmetric matrix ($A^T = -A$) such that A is 2019 x 2019

55/55

. since A is of odd size, its characteristic polynomial is of odd degree, such a polynomial has at least one real root, hence A has at least one real eigenvalue

Let α be the real eigenvalue of A, then $Ax = \alpha x$ for $x \neq 0$.

$$
\begin{aligned}
\text{So } \alpha\langle x,x\rangle &= \langle x, \bar{\alpha} x\rangle \quad \longrightarrow \text{ since } \alpha \text{ is real} \\
&= \langle x, \alpha x\rangle \\
&= \langle x, Ax\rangle \\
&= x^T A x \\
&= (A^T x)^T x \\
&= \langle A^T x, x\rangle \\
&= -\langle Ax, x\rangle \\
&= -\langle \alpha x, x\rangle \\
&= -\alpha\langle x, x\rangle
\end{aligned}
$$

5/5

since $\langle x,x\rangle \neq 0$ then $\alpha = -\alpha \Rightarrow 2\alpha = 0 \Rightarrow \alpha = 0$

thus A is not invertible

* Question 2:

5/5

Let $A = J_3^{(2)} \oplus J_2^{(2)} \oplus J_3^{(1)}$

then $C_A(x) = (x-3)^3 (x-2)^2$

and $m_A(x) = (x-3)^2 (x-2)^2$

thus $R_A = C(f_1) \oplus C(f_2) \oplus C(f_3)$

such that $f_1 = (x-3)^2 = x^2 - 6x + 9$

$f_2 = (x-2)^2 = x^2 - 4x + 4$

$f_3 = x - 3$

hence $R_A = \begin{bmatrix} 0 & -9 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

5/5

* Question 3: Let $T: V \rightarrow V$ be a linear tranformation and $F: V \rightarrow V$ such that $F = T^2 + 5T + 2019I$

. Let $W = Z(F)$

For every $\omega \in W$ we have:

$F(\omega) = 0$

hence $T^2(\omega) + 5T(\omega) + 2019\omega = 0$

$\Rightarrow T(T^2(\omega)) + 5T(T(\omega)) + 2019T(\omega) = T(0)$

$\Rightarrow T^2(T(\omega)) + 5T(T(\omega)) + 2019T(\omega) = 0$

$\Rightarrow F(T(\omega)) = 0$

$\Rightarrow T(\omega) \in Z(F)$

$\Rightarrow T(\omega) \in W$ for every $\omega \in W$

5/5

* Question 4:

Let $A = \begin{bmatrix} 3 & 6 & 3 \\ -3 & 0 & 3 \\ -3 & -6 & 0 \end{bmatrix}$

step 1: gcd (all enteries of A) = $d_1 = 3$

step 2: $|A| = |D| = -6 \begin{vmatrix} -3 & 3 \\ -3 & 0 \end{vmatrix} + 6 \begin{vmatrix} 3 & 3 \\ -3 & 3 \end{vmatrix}$

$= -6(+9) + 6(9+9)$

$= -54 + 108$

$= 54$

thus $d_1 = d_2 = 3$ and $d_3 = 6$

Hence $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

10/10

and $D = RAC$

$\longrightarrow$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}3&6&3\\-3&0&3\\-3&-6&0\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$R_1+R_2 \to R_2$
$R_1+R_3 \to R_3$

$$\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix} \quad \begin{bmatrix}3&6&3\\0&6&6\\0&0&3\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$-2C_1+C_2 \to C_2$
$-C_1+C_3 \to C_3$

$$\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix} \quad \begin{bmatrix}3&0&0\\0&6&6\\0&0&3\end{bmatrix} \quad \begin{bmatrix}1&-2&-1\\0&1&0\\0&0&1\end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix}1&0&0\\1&0&1\\1&1&0\end{bmatrix} \quad \begin{bmatrix}3&0&0\\0&0&3\\0&6&6\end{bmatrix} \quad \begin{bmatrix}1&-2&-1\\0&1&0\\0&0&1\end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix}1&0&0\\1&0&1\\1&1&0\end{bmatrix} \quad \begin{bmatrix}3&0&0\\0&3&0\\0&6&6\end{bmatrix} \quad \begin{bmatrix}1&-1&-2\\0&0&1\\0&1&0\end{bmatrix}$$

$-C_3+C_2 \to C_2$

$$\underset{R}{\underset{\shortparallel}{\begin{bmatrix}1&0&0\\1&0&1\\1&1&0\end{bmatrix}}} \quad \underset{D}{\underset{\shortparallel}{\begin{bmatrix}3&0&0\\0&3&0\\0&0&6\end{bmatrix}}} \quad \underset{C}{\underset{\shortparallel}{\begin{bmatrix}1&1&-2\\0&-1&1\\0&1&0\end{bmatrix}}}$$

and $D = RAC$

* **Question 5:** Let $A = \begin{bmatrix} 2 & 4 & 4 & 2 & 6 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

1) $C_A(x) = (x-2)^3 (x-3)^2$

2) By using online calculator: $m_A(x) = (x-2)^2 (x-3)^2$

3) We have two eigenvalues 2 and 3:

$IN(E_2) = 2$

$IN(E_3) = 1$

4) $J = J_2^{(2)} \oplus J_3^{(2)} \oplus J_2^{(1)}$

$$= \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

5/5

5) $R_A = C(f_1) \oplus C(f_2) \oplus C(f_3)$

where $f_1 = (x-2)^2 = x^2 - 4x + 4$

$f_2 = (x-3)^2 = x^2 - 6x + 9$

$f_3 = (x-2)$

thus $R_A = \begin{bmatrix} 0 & -4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

5/5

* Question 6:

1) suppose $m_A(X) = X^m + a_{m-1}X^{m-1} + \dots + a_1 X + a_0$

and $m_{A^T}(X) = X^n + b_{n-1}X^{n-1} + \dots + b_1 X + b_0$

. we know that $m_{A^T}(A^T) = 0$

and $m_A(A^T) = (A^T)^m + a_{m-1}(A^T)^{m-1} + \dots + a_1 A^T + a_0 I_m$

$= (A^m + a_{m-1}A^{m-1} + \dots + a_1 A + a_0 I_m)^T$ since $m_A(A) = 0$

$= 0$

thus $m_{A^T}(X)$ divides $m_A(X)$

and we know that $m_A(A) = 0$

and $m_{A^T}(A) = A^n + b_{n-1}A^{n-1} + \dots + b_1 A + b_0 I_n$

$= [(A^T)^n + b_{n-1}(A^T)^{n-1} + \dots + b_1 A^T + b_0 I_n]^T$ since $m_{A^T}(A^T) = 0$

$= 0$

thus $m_A(X)$ divides $m_{A^T}(X)$

therefore they must be equal

5/5

2) A is similar to B $\Rightarrow$ $A = QBQ^{-1}$

B is similar to C $\Rightarrow$ $B = PCP^{-1}$

then $A = Q(PCP^{-1})Q^{-1}$

$= QP\,C\,(QP)^{-1}$ $\rightarrow$ let $W = QP \Rightarrow W^{-1} = (QP)^{-1}$

$= W\,C\,W^{-1}$

therefore A and C are similar.

5/5

3) We know that $C_A(x) = C_{A^T}(x)$

we proved in (1) that $m_A(x) = m_{A^T}(x)$

Moreover, $IN(E_a)$ is the same for A and A^T

So we can conclude that $R_A = R_{A^T}$

5/5

and we know that A is similar to R_A

and $R_{A^T} = R_A$ is similar to A^T

thus by (2) A is similar to A^T.